

Derivation of a Convection Process in a Steady Diffusion-Transfer Problem by Homogenization*

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Abstract

We study the homogenization of a steady diffusion equation in a highly heterogeneous medium made of two subregions separated by a periodic barrier through which the flow is proportional to the jump of the temperature by a layer conductance of the same order of magnitude of the materials in consideration. The macroscopic governing equations and the effective conductivity of the homogenized model are obtained by means of the two scale convergence technique. We show that under some hypothesis the homogenized systems contain convective terms of order one.

1 Introduction

Homogenization in multicomponent ε -periodic media with interfacial barriers has been extensively studied these last years. There are many mathematical works devoted to the subject and we refer the reader for instance to Auriault et al [5], [6], Benveniste [8], Canon et al. [9], Hummel [11], Monsurró [14] and Pernin [17]. . .

In [5], [6] the layer conductance or sometimes called the resistivity is considered as a positive function of order of magnitude ε^γ and five distinct macroscopic models are derived by the formal asymptotic expansion method [7], [18]. These homogenized models are related to five values of γ which are $-2, -1, 0, 1$ and 2 . Monsurró [14] considered the same problem and the derivation of the homogenized models is obtained with the help of the oscillating test functions method of Tartar [19]. The case $\gamma \leq -1$ has been studied by Hummel [11] but for media with disconnected components arranged in a tessellation configuration. He

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used the two-scale convergence method by Nguetseng [16], further developed by Allaire [3] and adapted to periodic surfaces, see for e.g. Allaire et al [4] and Neuss-Radu [15]. The aim of this paper is to consider in the simplest configuration the case of resistivity having zero average value on the periodic interface. As in Ainouz [2] we show that the homogenized problem contains convective terms.

The paper is organized as follows: in Section 2 we give the geometry of the medium in which the stationary diffusion micro-model is set. In Section 3 the weak formulation of the problem is studied in a suitable function space and we give the main a priori estimates. Finally in Section 4, we recall the two-scale convergence and its main results, which we shall use it to derive the homogenized problem with the help of two-scale convergence method.

2 Setting of the Problem and statement of the main result

We begin this section by describing the geometry of the medium underlying the exact micro-model for the steady diffusion equation in a two-component periodic medium.

Let $Y = (0, 1)^n$ be the unit cell of periodicity and assume that Y is divided as $Y = Y_1 \cup Y_2 \cup \Sigma$ where $Y_i, i = 1, 2$ are two open subsets of Y and $\Sigma = \partial Y_1 \cap \partial Y_2$ is a sufficiently smooth interface that separates them. The sets Y_1 and Y_2 are made of two different materials but having conductance of same order of magnitude and Σ is a thin layer of very low conductance which constitutes a flow exchange barrier.

Let $A_i(y)$ ($i = 1, 2$) denotes the conductivity tensor of the material Y_i . We assume that $A_i(y)$ is a $n \times n$ Y -periodic matrix-valued function and continuous on \mathbb{R}^n such that

$$m_i |\lambda|^2 \leq A_i(y) \lambda \cdot \lambda \quad (2.1)$$

and

$$A_i(y) \lambda \cdot \eta \leq M_i |\lambda| |\eta| \quad (2.2)$$

for a.e. $y \in Y_i$ and for all $\lambda, \eta \in \mathbb{R}^n$, where m_i, M_i are positive constants.

Let α denotes the barrier resistivity of Σ . For simplicity we suppose that α is a continuous function on \mathbb{R}^n , Y -periodic with a zero average-value on Σ , i.e.

$$\int_{\Sigma} \alpha(y) d\sigma(y) = 0. \quad (2.3)$$

Here $d\sigma(y)$ denotes the surface measure on Σ . We decompose α into its positive and negative parts as follows:

$$\alpha = \alpha^+ - \alpha^-, \quad (2.4)$$

where

$$\alpha^+ = \sup(\alpha, 0), \quad \alpha^- = \sup(-\alpha, 0).$$

We assume that

$$\alpha^+ \geq \alpha_0 \tag{2.5}$$

where α_0 is a positive real number.

Let $a_i(y)$ ($i = 1, 2$) be continuous functions on \mathbb{R}^n and Y -periodic such that

$$a_i(y) \geq \eta_i > 0 \text{ for a.e. } y \in Y_i. \tag{2.6}$$

Let Ω be a bounded domain in \mathbb{R}^n with sufficiently smooth boundary Γ . Let $\varepsilon > 0$ be a positive number taking its value in a sequence of real numbers converging to zero. We consider the following open subsets of Ω

$$\Omega_i^\varepsilon = \{x \in \Omega; \chi_i\left(\frac{x}{\varepsilon}\right) = 1\} \quad i = 1, 2 \tag{2.7}$$

where $\chi_i(y)$ is the Y -periodic characteristic function of Y_i .

The $(n-1)$ -dimensional surface Σ^ε defined by

$$\Sigma^\varepsilon = \{x \in \Omega; \exists \vec{k} \in \mathbb{Z}^n, \frac{x}{\varepsilon} + \vec{k} \in \Sigma\} \tag{2.8}$$

is then by construction the interfacial barrier that separates the materials Ω_1^ε and Ω_2^ε , i.e. $\Sigma^\varepsilon = \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon$.

Let $\Gamma_i^\varepsilon = \overline{\Omega_i^\varepsilon} \cap \Gamma$ ($i = 1, 2$) and assume that $|\Gamma_i^\varepsilon| \neq 0$. Let $f_i^\varepsilon \in L^2(\Omega_i^\varepsilon)$ ($i = 1, 2$) be uniformly bounded functions, that is,

$$\|f_i^\varepsilon\|_{0, \Omega_i^\varepsilon} \leq C \tag{2.9}$$

where (and throughout this paper) C is a positive constant independent of ε .

Let us consider the following transmission problem:

$$-\operatorname{div}\left(A_i\left(\frac{x}{\varepsilon}\right)\nabla u_i^\varepsilon(x)\right) + a_i\left(\frac{x}{\varepsilon}\right)u_i^\varepsilon(x) = f_i^\varepsilon(x) \text{ in } \Omega_i^\varepsilon, \quad i = 1, 2 \tag{2.10}$$

$$A_1\left(\frac{x}{\varepsilon}\right)\nabla u_1^\varepsilon(x) \cdot \nu^\varepsilon = A_2\left(\frac{x}{\varepsilon}\right)\nabla u_2^\varepsilon(x) \cdot \nu^\varepsilon \text{ on } \Sigma^\varepsilon, \tag{2.11}$$

$$A_1\left(\frac{x}{\varepsilon}\right)\nabla u_1^\varepsilon(x) \cdot \nu^\varepsilon = -\alpha\left(\frac{x}{\varepsilon}\right)(u_1^\varepsilon - u_2^\varepsilon) \text{ on } \Sigma^\varepsilon, \tag{2.12}$$

$$u_i^\varepsilon = 0 \text{ on } \Gamma_i^\varepsilon, \quad i = 1, 2 \tag{2.13}$$

where ν^ε is the unit outward normal to Ω_1^ε obtained in an obvious way by extending ν , the unit outward normal to Y_1 , by Y -periodicity to the whole space \mathbb{R}^n .

The first equation (2.10) is the classical steady diffusion problem. The second equation (2.11) expresses the continuity of the flow across the interfacial contact Σ^ε while the boundary condition (2.12) says that the flow is proportional to the jump of the temperature. For a physical justification of the condition (2.12) we refer the reader to Carslaw and Jaeger [10], Kholodovskii [12]. Finally the last condition (2.13) is the well-known Dirichlet boundary condition.

In order to state the main theorem of this paper we begin by setting some notations. Let us first consider the following microscopic problems:

$$-\operatorname{div}_y(A_i(y)(\nabla_y \gamma_i(y))) = 0 \text{ in } Y_i, \quad (2.14)$$

$$(A_i(y)(\nabla_y \gamma_i(y))) \cdot \nu(y) = -\alpha(y) \text{ on } \Sigma, \quad (2.15)$$

$$y \longmapsto \gamma_i(y) \text{ } Y\text{-periodic,} \quad i = 1, 2, \quad (2.16)$$

and for $k = 1, 2, \dots, n$ the problem

$$-\operatorname{div}_y(A_i(y)(e^k + \nabla_y \xi_i^k(y))) = 0 \text{ in } Y_i, \quad (2.17)$$

$$(A_i(y)(e^k + \nabla_y \xi_i^k(y))) \cdot \nu(y) = 0 \text{ on } \Sigma, \quad (2.18)$$

$$y \longmapsto \xi_i^k(y) \text{ } Y\text{-periodic,} \quad i = 1, 2, . \quad (2.19)$$

where div_y (resp. ∇_y) is the divergence (resp. gradient) operator with respect to the variable y and $(e^k)_{1 \leq k \leq n}$ is the canonical basis of \mathbb{R}^n .

The well-posedness of these elliptic problems is well-known, see for instance Bensoussan et al. [7]. Let us mention that the assumption (2.3) is a compatibility condition for the unique solvability of the problems (2.14)-(2.16).

Let us define the effective matrices $A_i^{eff} = (a_i^{kj})_{1 \leq k, j \leq n}$ and the convection vectors $B_i = (b_i^k)_{1 \leq k \leq n}$ ($i = 1, 2$)

$$a_i^{kj} = \int_{Y_i} A_i(e^k + \nabla_y \xi_i^k) \cdot (e^j + \nabla_y \xi_i^j) dy, \quad (2.20)$$

$$b_i^k = (-1)^{i-1} \int_{Y_i} A_i \nabla \gamma_i dy + \int_{\Sigma} \alpha \xi_i^k d\sigma(y). \quad (2.21)$$

Let us denote

$$d = \int_{\Sigma} \alpha(\gamma_1 - \gamma_2) d\sigma(y) \quad (2.22)$$

and

$$c_i = d + \int_{Y_i} a_i dy, \quad g_i(x) = \int_{Y_i} f_i(x, y) dy, \quad i = 1, 2 \quad (2.23)$$

where f_i is the two-scale limit of the sequence $(\chi_i(\frac{x}{\varepsilon}) f_i^\varepsilon)_{\varepsilon > 0}$.

The main result of the paper is the following theorem

Theorem 2.1 *Let $(u_1^\varepsilon, u_2^\varepsilon)$ be the solution of the problem (2.10)-(2.13). Then up to a subsequence, $(\chi_1(\frac{x}{\varepsilon})u_1^\varepsilon, \chi_2(\frac{x}{\varepsilon})u_2^\varepsilon)_{\varepsilon>0}$ two-scale converges to $(u_1, u_2) \in (H_0^1(\Omega))^2$ solution of the Homogenized Problem:*

$$-\operatorname{div}(A_1^{eff} \nabla u_1) + B_1 \cdot \nabla u_1 - B_2 \cdot \nabla u_2 + c_1 u_1 - d u_2 = g_1 \text{ in } \Omega, \quad (2.24)$$

$$-\operatorname{div}(A_2^{eff} \nabla u_2) + B_2 \cdot \nabla u_2 - B_1 \cdot \nabla u_1 + c_2 u_2 - d u_1 = g_2 \text{ in } \Omega, \quad (2.25)$$

$$u_1 = u_2 = 0 \text{ on } \Gamma. \quad (2.26)$$

3 Solvability of the problem and a priori estimates

Let us introduce the Hilbert space $V^\varepsilon = H^1(\Omega_1^\varepsilon, \Gamma_1^\varepsilon) \times H^1(\Omega_2^\varepsilon, \Gamma_2^\varepsilon)$ where

$$H^1(\Omega_i^\varepsilon, \Gamma_i^\varepsilon) = \{v_i \in H^1(\Omega_i^\varepsilon); v_i = 0 \text{ on } \Gamma_i^\varepsilon\}, \quad i = 1, 2 \quad (3.1)$$

V^ε is equipped with the norm:

$$\|(v_1, v_2)\|_{V^\varepsilon}^2 := \|v_1\|_{1, \Omega_1^\varepsilon}^2 + \|v_2\|_{1, \Omega_2^\varepsilon}^2 + \|v_1 - v_2\|_{0, \Sigma^\varepsilon}^2. \quad (3.2)$$

The equivalent weak formulation of the micro-model (2.10)-(2.13) is :

$$\begin{aligned} &\text{For each } \varepsilon > 0, \text{ find } (u_1^\varepsilon, u_2^\varepsilon) \in V^\varepsilon \text{ such that} \\ &a^\varepsilon((u_1^\varepsilon, u_2^\varepsilon), (v_1, v_2)) = L^\varepsilon((v_1, v_2)) \text{ for all } (v_1, v_2) \in V^\varepsilon \end{aligned} \quad (3.3)$$

where for all $(w_1, w_2), (v_1, v_2) \in V^\varepsilon$

$$\begin{aligned} a^\varepsilon((w_1, w_2), (v_1, v_2)) &= \int_{\Omega} (A(\frac{x}{\varepsilon}) \nabla w^\varepsilon \cdot \nabla v^\varepsilon + a(\frac{x}{\varepsilon}) w^\varepsilon v^\varepsilon) dx + \\ &+ \int_{\Sigma^\varepsilon} \alpha(\frac{x}{\varepsilon}) (w_1 - w_2)(v_1 - v_2) d\sigma^\varepsilon(x), \end{aligned} \quad (3.4)$$

$$L^\varepsilon((v_1, v_2)) = \left(\int_{\Omega} f^\varepsilon(x) v^\varepsilon(x) dx \right). \quad (3.5)$$

Here $d\sigma^\varepsilon(x)$ denotes the surface measure on Σ^ε and $A, w^\varepsilon, v^\varepsilon, f^\varepsilon$ are defined by

$$A(y) = \chi_1(y) A_1(y) + \chi_2(y) A_2(y), \quad y \in \mathbb{R}^n,$$

$$w^\varepsilon = \chi_1(\frac{x}{\varepsilon}) w_1(x) + \chi_2(\frac{x}{\varepsilon}) w_2(x), \quad x \in \Omega,$$

$$v^\varepsilon = \chi_1(\frac{x}{\varepsilon}) v_1(x) + \chi_2(\frac{x}{\varepsilon}) v_2(x), \quad x \in \Omega,$$

$$f^\varepsilon = \chi_1(\frac{x}{\varepsilon}) f_1^\varepsilon(x) + \chi_2(\frac{x}{\varepsilon}) f_2^\varepsilon(x), \quad x \in \Omega.$$

To study the solvability of the problem (3.3) we shall use the following inequality:

Lemma 3.1 *There exists a constant $C_i > 0$ independent of ε such that for every $v_i \in H^1(\Omega_i^\varepsilon)$ and for all $\delta_i > 0$ we have*

$$\|v_i\|_{0,\Sigma^\varepsilon}^2 \leq C_i \left(\frac{1}{\delta_i \varepsilon} \|v_i\|_{0,\Omega_i^\varepsilon}^2 + \delta_i \varepsilon \|\nabla v_i\|_{0,\Omega_i^\varepsilon}^2 \right). \quad i = 1, 2 \quad (3.6)$$

Proof. See for instance Ainouz [2] (see also Monsurro [14]). Note that the constants C_i depends solely on the geometry of Y_i . ■

Throughout this paper we assume that

$$\|\alpha^-\|_{\infty,\Sigma^\varepsilon} \leq \min_{i=1,2} \frac{\sqrt{m_i \eta_i}}{C_i}. \quad (3.7)$$

where C_i are the constants defined in Lemma 3.1. Now we give the following existence result :

Proposition 3.2 *Let the assumption (3.7) be fulfilled. Then for any fixed $\varepsilon > 0$, there exists a unique couple $(u_1^\varepsilon, u_2^\varepsilon) \in V^\varepsilon$ solution of (3.3). Moreover we have the a priori estimate*

$$\|(u_1^\varepsilon, u_2^\varepsilon)\|_{V^\varepsilon} \leq C. \quad (3.8)$$

Proof. First we show that $a^\varepsilon(\cdot, \cdot)$ is coercive on V^ε . Let $(v_1, v_2) \in V^\varepsilon$, then

$$\begin{aligned} a^\varepsilon((v_1, v_2), (v_1, v_2)) &\geq \sum_{i=1,2} (m_i \|\nabla v_i\|_{0,\Omega_i^\varepsilon}^2 + \eta_i \|v_i\|_{0,\Omega_i^\varepsilon}^2) \\ &\quad + (\alpha_0 - \|\alpha^-\|_{\infty,\Sigma^\varepsilon}) \int_{\Sigma^\varepsilon} (v_1(x) - v_2(x))^2 d\sigma^\varepsilon(x). \end{aligned} \quad (3.9)$$

But in view of Lemma 3.1 we see that

$$\int_{\Sigma^\varepsilon} (v_1(x) - v_2(x))^2 d\sigma^\varepsilon(x) \leq \sum_{i=1,2} C_i \left(\frac{1}{\delta_i \varepsilon} \|v_i\|_{0,\Omega_i^\varepsilon}^2 + \delta_i \varepsilon \|\nabla v_i\|_{0,\Omega_i^\varepsilon}^2 \right).$$

Set

$$\beta_i = m_i - C_i \delta_i \varepsilon \|\alpha^-\|_{\infty,\Sigma^\varepsilon}, \quad \gamma_i := \eta_i - \frac{C_i}{\delta_i \varepsilon} \|\alpha^-\|_{\infty,\Sigma^\varepsilon}.$$

Therefore (3.9) becomes

$$\begin{aligned} a^\varepsilon((v_1, v_2), (v_1, v_2)) &\geq \sum_{i=1,2} \left(\beta_i \|\nabla v_i\|_{0,\Omega_i^\varepsilon}^2 + \gamma_i \|v_i\|_{0,\Omega_i^\varepsilon}^2 \right) \\ &\quad + \alpha_0 \|v_1 - v_2\|_{0,\Sigma^\varepsilon}^2. \end{aligned} \quad (3.10)$$

Now we choose, for example, $\delta_i = \frac{2\varepsilon^{-1} C_i m_i \|\alpha^-\|_{\infty,\Sigma^\varepsilon}}{\eta_i m_i + (C_i \|\alpha^-\|_{\infty,\Sigma^\varepsilon})^2}$, $i = 1, 2$. Then by (3.7) we have

$$\beta_i = \frac{m_i (\eta_i m_i - (C_i \|\alpha^-\|_{\infty,\Sigma^\varepsilon})^2)}{\eta_i m_i + (C_i \|\alpha^-\|_{\infty,\Sigma^\varepsilon})^2} > 0, \quad (3.11)$$

and

$$\gamma_i = \frac{\eta_i m_i - (C_i \|\alpha^-\|_{\infty, \Sigma^\varepsilon})^2}{2C_i m_i \|\alpha^-\|_{\infty, \Sigma^\varepsilon}} > 0. \quad (3.12)$$

Hence by (3.11) and (3.12), the inequality (3.10) becomes

$$a^\varepsilon((v_1, v_2), (v_1, v_2)) \geq c_0 \|(v_1, v_2)\|_{V^\varepsilon}^2 \quad (3.13)$$

where

$$c_0 = \min(\beta_1, \beta_2, \gamma_1, \gamma_2, \alpha_0) > 0$$

which is independent of ε .

It is easy to observe that $a^\varepsilon((w_1, w_2), (v_1, v_2))$ is bilinear continuous on $V^\varepsilon \times V^\varepsilon$ and that $L^\varepsilon((v_1, v_2))$ is linear and continuous on V^ε . Consequently by the Lax-Milgram Lemma the problem (3.3) has a unique solution $(u_1^\varepsilon, u_2^\varepsilon)$ in V^ε . Furthermore we have

$$\|(u_1^\varepsilon, u_2^\varepsilon)\|_{V^\varepsilon}^2 \leq \frac{1}{c_0} L^\varepsilon((u_1^\varepsilon, u_2^\varepsilon)) \leq \frac{1}{c_0} \sum_{i=1,2} \|f_i^\varepsilon\|_{0, \Omega_i^\varepsilon} \|u_i^\varepsilon\|_{0, \Omega_i^\varepsilon}$$

which implies by (2.9) that the sequence $(u_1^\varepsilon, u_2^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in V^ε . The Proposition is then proved. \blacksquare

In view of the a priori estimate (3.8), one is interested in investigating the limit of the sequence $((u_1^\varepsilon, u_2^\varepsilon))_{\varepsilon>0}$ as $\varepsilon \rightarrow 0$ in a sense that will be specified later. This is the purpose of the next Section.

4 two-scale convergence process

In this Section we shall use the two-scale convergence method to determine the Homogenization of the Problem (2.10)-(2.13). For more details on two-scale convergence, we refer the reader to Nguesteng [16], Allaire [3] and a recent paper by Lukkassen et al. [13]. We also mention the work by Allaire et al. [4] (see also Ainouz [1], Neuss-radu [15]) where the two-scale convergence is applied to periodic surfaces. For convenience we recall the definition and the main compactness results of this method.

Definition 4.1

1. Let $(v^\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(\Omega)$ and $v_0 \in L^2(\Omega \times Y)$. We say that $(v^\varepsilon)_{\varepsilon>0}$ two-scale converges to v_0 if for every $\varphi \in \mathcal{D}(\Omega; \mathcal{C}_{per}^\infty(Y))$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v^\varepsilon(x) \varphi(x, \frac{x}{\varepsilon}) dx = \int_{\Omega \times Y} v_0(x, y) \varphi(x, y) dx dy. \quad (4.1)$$

2. Let $(v^\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(\Sigma^\varepsilon)$ and $v_0 \in L^2(\Omega \times \Sigma)$. We say that $(v^\varepsilon)_{\varepsilon>0}$ two-scale converges to v_0 if for every $\varphi \in \mathcal{D}(\bar{\Omega}; \mathcal{C}_{per}^\infty(Y))$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma^\varepsilon} v^\varepsilon(x) \varphi(x, \frac{x}{\varepsilon}) d\sigma^\varepsilon(x) = \int_{\Omega \times \Sigma} v_0(x, y) \varphi(x, y) dx d\sigma(y). \quad (4.2)$$

We now give the main compactness results of the two scale convergence :

Theorem 4.2

1. Let $(v^\varepsilon)_{\varepsilon>0}$ be a uniformly bounded sequence in $L^2(\Omega)$. Then one can extract a subsequence which two-scale converges to a function $v_0 \in L^2(\Omega \times Y)$ in the sense of Definition 4.11.
2. Let $(\sqrt{\varepsilon} v^\varepsilon)_{\varepsilon>0}$ be a uniformly bounded sequence in $L^2(\Sigma^\varepsilon)$. Then one can extract a subsequence which two-scale converges to a function $w_0 \in L^2(\Omega \times \Sigma)$.
3. If $(v^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) then one can extract a subsequence still denoted $(v^\varepsilon)_{\varepsilon>0}$ and there exist $v \in H^1(\Omega)$ (resp. $H_0^1(\Omega)$) and $V \in L^2(\Omega; H_{per}^1(Y)/\mathbb{R})$ such that
 - i) $(v^\varepsilon)_{\varepsilon>0}$ two-scale converges to v in the sense of Definition 4.11.
 - ii) $(\nabla v^\varepsilon)_{\varepsilon>0}$ two-scale converges to $\nabla v + \nabla_y V$ in the sense of Definition 4.11.
 - iii) Moreover $(\sqrt{\varepsilon} v^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(\Sigma^\varepsilon)$ and $(v^\varepsilon)_{\varepsilon>0}$ two-scale converges to v in the sense of Definition 4.12. we have $w_0 = v_0$.

Next we shall apply these results to determine the two-scale limit of $(u_i^\varepsilon)_{\varepsilon>0}$ and the source terms $(f_i^\varepsilon)_{\varepsilon>0}$, $i = 1, 2$.

Lemma 4.3 For each $i = 1, 2$ there exist $f_i \in L^2(\Omega \times Y_i)$, $u_i(x) \in H_0^1(\Omega)$ and $U_i \in L^2(\Omega, H_{per}^1(Y_i)/\mathbb{R})$ such that up to a subsequence we have for all $\varphi_i \in \mathcal{D}(\Omega; \mathcal{C}_{per}^\infty(Y_i))$ and $\psi_i \in \mathcal{D}(\Omega; \mathcal{C}_{per}^\infty(Y_i))^n$:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_i^\varepsilon} f_i^\varepsilon(x) \varphi_i(x, \frac{x}{\varepsilon}) dx = \int_{\Omega} \int_{Y_i} f_i(x, y) \varphi_i(x, y) dy dx, \quad (4.3)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_i^\varepsilon} u_i^\varepsilon(x) \varphi_i(x, \frac{x}{\varepsilon}) dx = \int_{\Omega} \int_{Y_i} u_i(x) \varphi_i(x, y) dy dx, \quad (4.4)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla u_i^\varepsilon(x) \psi_i(x, \frac{x}{\varepsilon}) dx = \int_{\Omega} \int_{Y_i} [\nabla u_i(x) + \nabla_y U_i(x, y)] \psi_i(x, y) dy dx \quad (4.5)$$

and for all $\varphi \in \mathcal{D}(\bar{\Omega}; \mathcal{C}_{per}^\infty(Y))$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma^\varepsilon} \varepsilon u_i^\varepsilon(x) \varphi(x, \frac{x}{\varepsilon}) d\sigma^\varepsilon(x) = \int_{\Omega} \int_{\Sigma} u_i(x) \varphi(x, y) d\sigma(y) dx, \quad i = 1, 2. \quad (4.6)$$

Proof. The existence of the two-scale limits is an immediate consequence of the a priori estimates (2.9), (3.8) and the definition of two-scale convergence. Indeed, since for $i = 1, 2$ the sequences $\chi_i(\frac{x}{\varepsilon})f_i^\varepsilon(x)$, $\chi_i(\frac{x}{\varepsilon})u_i^\varepsilon(x)_{\varepsilon>0}$, $(\chi_i(\frac{x}{\varepsilon})\nabla u_i^\varepsilon(x))_{\varepsilon>0}$ and $(\sqrt{\varepsilon}u_i^\varepsilon(x))_{\varepsilon>0}$ are uniformly bounded in $L^2(\Omega)$, $L^2(\Omega)$, $(L^2(\Omega))^n$ and $L^2(\Sigma^\varepsilon)$ respectively. Then by Theorem 4.2, one can extract a subsequence still denoted ε and there exist $f_i(x, y)$, $u_i^0(x, y) \in L^2(\Omega \times Y)$ and $\xi_i \in (L^2(\Omega \times Y))^n$ such that $\chi_i(\frac{x}{\varepsilon})f_i^\varepsilon(x)$, $\chi_i(\frac{x}{\varepsilon})u_i^\varepsilon(x)$ and $\chi_i(\frac{x}{\varepsilon})\nabla u_i^\varepsilon(x)$ two-scale converge respectively to $f_i(x, y)$, $u_i(x, y)$ and $\xi_i(x, y)$. We point out that $f_i(x, y)$, $u_i(x, y)$ and $\xi_i(x, y)$ are equal zero outside Y_i . Arguing as in Allaire [3, Thm 2.9] we easily arrive at $u_i^0(x, y) = \chi_i(y)u_i(x)$ where $u_i(x) \in H_0^1(\Omega)$ and $\xi_i(x, y) = \nabla u_i(x) + \nabla_y U_i(x, y)$ where $U_i \in L^2(\Omega, H_{per}^1(Y_i)/\mathbb{R})$. Finally, by Allaire et al. [4, Prop. 2.6] (see also Ainouz [1] or Neuss-Radu [15, Thm. 2.2]) we obtain the last limit. ■

The following result is an extension of an auxiliary result given in [2].

Lemma 4.4 *Up to the subsequence given in Lemma 4.3, we have*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma^\varepsilon} u_i^\varepsilon(x) \varphi_i(x, \frac{x}{\varepsilon}) d\sigma^\varepsilon(x) = \int_{\Omega} \int_{\Sigma} U_i(x, y) \varphi_i(x, y) d\sigma(y) dx$$

for any $\varphi_i \in \mathcal{D}(\overline{\Omega}; \mathcal{C}_{per}(\overline{Y_i}))$ such that $\int_{\Sigma} \varphi_i(x, y) d\sigma(y) = 0 \quad i = 1, 2$.

Proof. For a fixed x in $\overline{\Omega}$, and $i = 1, 2$ let $\phi_i(x, y)$ be the solution of the following boundary value problem

$$-\operatorname{div}_y \phi_i(x, y) = 0 \text{ in } Y_i \quad (4.7)$$

$$\phi_i(x, y) \cdot \nu(y) = (-1)^{i-1} \varphi_i(x, y) \text{ on } \Sigma, \quad (4.8)$$

$$y \mapsto \phi_i(x, y) \text{ is } Y\text{-periodic.} \quad (4.9)$$

Such a function exists since $\int_{\Sigma} \varphi_i(x, y) d\sigma(y) = 0$. Furthermore, the solution $\phi_i(x, \cdot)$ belongs to $H_{per}^1(Y_i)/\mathbb{R}$. Taking into account the boundary conditions (4.8) and (4.9) we see that

$$\int_{\Sigma^\varepsilon} u_i^\varepsilon(x) \varphi_i(x, y) d\sigma^\varepsilon(x) = \int_{\Omega_i^\varepsilon} \nabla u_i^\varepsilon(x) \phi_i(x, \frac{x}{\varepsilon}) dx + \int_{\Omega_i^\varepsilon} u_i^\varepsilon(x) \operatorname{div}_x(\phi_i(x, \frac{x}{\varepsilon})) dx. \quad (4.10)$$

By (4.7) and letting $\varepsilon \rightarrow 0$ in (4.10) together with (4.4), (4.5) we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Sigma^\varepsilon} u_i^\varepsilon(x) \varphi_i(x, y) d\sigma^\varepsilon(x) &= \int_{\Omega} \int_{Y_i} [\nabla u_i(x) + \nabla_y U_i(x, y)] \phi_i(x, y) dy dx \\ &+ \int_{\Omega} \int_{Y_i} u_i(x) \operatorname{div}_x \phi_i(x, y) dy dx. \end{aligned} \quad (4.11)$$

Since $u_i \in H_0^1(\Omega)$, integration by parts with respect to x gives

$$\int_{\Omega} \nabla u_i(x) \phi_i(x, y) dx + \int_{\Omega} u_i(x) \operatorname{div}_x \phi_i(x, y) dx = 0$$

Therefore (4.11) becomes

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma^\varepsilon} u_i^\varepsilon(x) \varphi_i(x, y) d\sigma^\varepsilon(x) = \int_{\Omega} \int_{Y_i} \nabla_y U_i(x, y) \phi_i(x, y) dy dx \quad (4.12)$$

Again, integrating by parts with respect to y in the right hand side of (4.12) and using (4.8), (4.9) yield

$$\begin{aligned} \int_{Y_i} \nabla_y U_i(x, y) \phi_i(x, y) dy dx &= - \int_{Y_i} U_i(x, y) \operatorname{div}_y \phi_i(x, y) dy dx + \\ &\quad (-1)^{i-1} \int_{\Sigma} U_i(x, y) (\phi_i(x, y) \cdot \nu(y)) d\sigma(y) dx \\ &= \int_{\Sigma} U_i(x, y) \varphi_i(x, y) d\sigma(y) dx. \end{aligned} \quad (4.13)$$

Finally, by combining (4.12) and (4.13) we arrive at the desired result. This proves the Lemma. \blacksquare

Now we are in a position to give the two-scale homogenized problem.

Proposition 4.5 *The two-scale limit $(u_i, U_i) \in H_0^1(\Omega) \times L^2(\Omega, H_{per}^1(Y_i)/\mathbb{R})$ is the solution of the two-scale homogenized system*

$$\left\{ \begin{array}{l} -\operatorname{div}_y(A_i(y)(\nabla u_i(x) + \nabla_y U_i(x, y))) = 0 \text{ in } \Omega \times Y_i, \quad i = 1, 2 \\ -\operatorname{div}_x(\int_{Y_i} A_i(\nabla u_i + \nabla_y U_i) dy) + \int_{Y_i} a_i(y) u_i(y) dy + \\ + (-1)^{i-1} \int_{\Sigma} \alpha(U_1 - U_2) d\sigma(y) = g_i \text{ in } \Omega, \quad i = 1, 2 \\ (A_1(\nabla u_1 + \nabla_y U_1)) \cdot \nu = (A_2(\nabla u_2 + \nabla_y U_2)) \cdot \nu \text{ on } \Omega \times \Sigma, \\ (A_1(\nabla u_1 + \nabla_y U_1)) \cdot \nu = -\alpha(u_1 - u_2) = 0 \text{ on } \Omega \times \Sigma, \\ u_i = 0 \text{ on } \Gamma, y \mapsto U_i(x, y) \text{ is } Y\text{-periodic.} \quad i = 1, 2 \end{array} \right. \quad (4.14)$$

Proof. Let $\varphi_i(x) \in \mathcal{D}(\Omega)$, $\Phi_i \in \mathcal{D}(\Omega, C_{per}^\infty(Y_i))$ $i = 1, 2$. We take $v_i(x) = \varphi_i(x) + \varepsilon \Phi_i(x, \frac{x}{\varepsilon})$ in the weak formulation (3.3). We have

$$\begin{aligned} a^\varepsilon((u_1^\varepsilon, u_2^\varepsilon), (v_1, v_2)) &= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon + \varepsilon J_1^\varepsilon + \varepsilon J_2^\varepsilon \\ L^\varepsilon((v_1, v_2)) &= K_1^\varepsilon + K_2^\varepsilon + \varepsilon L_1^\varepsilon + \varepsilon L_2^\varepsilon \end{aligned}$$

where

$$I_i^\varepsilon = \int_{\Omega_i^\varepsilon} \left\{ A_i\left(\frac{x}{\varepsilon}\right) \nabla u_i^\varepsilon(x) (\nabla \varphi_i(x) + \nabla_y \Phi_i(x, \frac{x}{\varepsilon})) + a_i\left(\frac{x}{\varepsilon}\right) u_i^\varepsilon(x) \varphi_i(x) \right\} dx, \quad i = 1, 2$$

$$\begin{aligned}
I_3^\varepsilon &= \int_{\Sigma^\varepsilon} \varepsilon \alpha\left(\frac{x}{\varepsilon}\right) (u_1^\varepsilon(x) - u_2^\varepsilon(x)) (\Phi_1(x, \frac{x}{\varepsilon}) - \Phi_2(x, \frac{x}{\varepsilon})) d\sigma^\varepsilon(x), \\
J_i^\varepsilon &= \int_{\Omega_i^\varepsilon} A_i\left(\frac{x}{\varepsilon}\right) \nabla u_i^\varepsilon(x) \nabla_x \Phi_i(x, \frac{x}{\varepsilon}) dx, \quad i = 1, 2 \\
I_4^\varepsilon &= \int_{\Sigma^\varepsilon} \alpha\left(\frac{x}{\varepsilon}\right) (u_1^\varepsilon(x) - u_2^\varepsilon(x)) (\varphi_1(x) - \varphi_2(x)) d\sigma^\varepsilon(x);
\end{aligned}$$

and

$$K_i^\varepsilon = \int_{\Omega_i^\varepsilon} f_i^\varepsilon(x) \varphi_i(x) dx; \quad L_i^\varepsilon = \int_{\Omega_i^\varepsilon} f_i^\varepsilon(x) \Phi_i(x, \frac{x}{\varepsilon}) dx, \quad i = 1, 2.$$

Letting $\varepsilon \rightarrow 0$ and using (4.3)-(4.6), we have

$$\begin{aligned}
\lim I_i^\varepsilon &= \int_{\Omega \times Y_i} \{A_i(y) (\nabla u_i(x) + \nabla_y U_i(x, y)) (\nabla \varphi_i(x) + \nabla_y \Phi_i(x, y)) dx dy\} + \\
&\quad \int_{\Omega \times Y_i} a_i(y) u_i(x) \varphi_i(x) dx dy, \quad i = 1, 2
\end{aligned} \tag{4.15}$$

$$\lim I_3^\varepsilon = \int_{\Omega \times \Sigma} \alpha(y) (u_1(x) - u_2(x)) (\Phi_1(x, y) - \Phi_2(x, y)) dx d\sigma(y) \tag{4.16}$$

$$\lim K_i^\varepsilon = \int_{\Omega \times Y_i} f_i(x, y) \varphi_i(x) dx dy; \quad i = 1, 2 \tag{4.17}$$

By virtue of Lemma 4.4

$$\lim I_4^\varepsilon = \int_{\Sigma^\varepsilon} \alpha(y) (U_1(x, y) - U_2(x, y)) (\varphi_1(x) - \varphi_2(x)) dx d\sigma(y); \tag{4.18}$$

On the other hand we see that

$$|J_i^\varepsilon| + |L_i^\varepsilon| \leq C, \quad i = 1, 2$$

where C is a positive constant independent of ε . Hence

$$\lim \varepsilon J_i^\varepsilon = \lim \varepsilon L_i^\varepsilon = 0, \quad i = 1, 2 \tag{4.19}$$

Now passing to the limit in (3.3) and using (4.15)-(4.19), we obtain the two-scale system

$$\begin{aligned}
&\sum_{i=1,2} \int_{\Omega \times Y_i} \{A_i(\nabla u_i + \nabla_y U_i)(\nabla \varphi_i + \nabla_y \Phi_i) + a_i u_i \varphi_i\} dx dy + \\
&+ \int_{\Omega \times \Sigma} \{\alpha(U_1 - U_2)(\varphi_1 - \varphi_2) + \alpha(u_1 - u_2)(\Phi_1 - \Phi_2)\} dx d\sigma(y) \\
&= \sum_{i=1,2} \int_{\Omega \times Y_i} f_i \varphi_i dx dy
\end{aligned} \tag{4.20}$$

Now by density the formulation (4.20) remains true for all (v_1, v_2, V_1, V_2) in $(H_0^1(\Omega))^2 \times L^2(\Omega, H_{per}^1(Y_i)/\mathbb{R}) \times L^2(\Omega, H_{per}^1(Y_2)/\mathbb{R})$. An integration by parts yields the two-scale homogenized system (4.14). ■

Proof of Theorem 2.1. First we take $\Phi_1, \varphi_2, \Phi_2 \equiv 0$ in (4.20). Then integration by parts yields

$$\begin{cases} -\operatorname{div}_x \left[\int_{Y_1} A_1(y) (\nabla u_1(x) + \nabla_y U_1(x, y)) dy \right] + \int_{Y_1} a_1(y) u_1(x) dy \\ + \int_{\Sigma} \alpha(y) (U_1(x, y) - U_2(x, y)) d\sigma(y) = \int_{Y_1} f_1(x, y) dy \text{ in } \Omega, \\ u_1 = 0 \text{ on } \Gamma. \end{cases} \quad (4.21)$$

Similarly we take $\varphi_1, \Phi_1, \Phi_2$ equal to zero and this gives

$$\begin{cases} -\operatorname{div}_x \left[\int_{Y_1} A_1(y) (\nabla u_1(x) + \nabla_y U_1(x, y)) dy \right] + \int_{Y_1} a_1(y) u_1(x) dy \\ + \int_{\Sigma} \alpha(y) (U_1(x, y) - U_2(x, y)) d\sigma(y) = \int_{Y_1} f_1(x, y) dy \text{ in } \Omega, \\ u_1 = 0 \text{ on } \Gamma. \end{cases} \quad (4.22)$$

Now choosing $\varphi_1, \varphi_2, \Phi_2 = 0$ (resp. $\varphi_1, \varphi_2, \Phi_1 = 0$) in (4.20) gives after integration by parts

$$\begin{cases} -\operatorname{div}_y (A_i(y) (\nabla u_i(x) + \nabla_y U_i(x, y))) = 0 \text{ in } \Omega \times Y_i, \\ (A_1(y) (\nabla u_1(x) + \nabla_y U_1(x, y))) \cdot \nu_1(y) \\ = (A_2(y) (\nabla u_2(x) + \nabla_y U_2(x, y))) \cdot \nu_1(y) \text{ on } \Omega \times \Sigma, \\ (A_1(y) (\nabla u_1(x) + \nabla_y U_1(x, y))) \cdot \nu_1(y) + \alpha(y) (u_1(x) - u_2(x)) = 0 \text{ on } \Omega \times \Sigma, \\ y \mapsto U_1(x, y), U_2(x, y) \text{ } Y\text{-periodic.} \end{cases} \quad (4.23)$$

■

Next we shall decouple the problem (4.23), that is eliminating the unknowns U_1, U_2 from the system (4.14). The linearity of the problem and the fact that u_i do not depend on the fast variable y enable us to put

$$U_i(x, y) = \sum_{k=1}^n \chi_i^k(y) \frac{\partial u_i}{\partial x_k}(x) + \gamma_i(y) (u_1(x) - u_2(x)) + \tilde{u}_i(x), \quad i = 1, 2 \quad (4.24)$$

Then inserting (4.24) into (4.21) and (4.22) yields the equations (2.24)-(2.26). Thus we have proved the Theorem 2.1

Remark 4.6 *It is easy to see that if A_i are symmetric for all i then $B_i = 0$.*

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